Finding Fair Full-State Feedback

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***Abstract*—The objective of this report is to design a full-state feedback controller to evaluate the control and determine the optimal system performance gains of a Furuta pendulum system using two primary method, manual tuning and linear quadratic regulator modeling. With the pre-determined state-space model obtained from the previous lab report, the performance was assessed by investigating the platen and pendulum rotations, the motor control effort, and its tracking errors. Several iterations of these tests were conducted, with the best performance coming from the manual tuning of all the gains on a downward pendulum configuration receiving 0.25Hz square wave inputs to maintain ±60° (Task 2). The resulting gains of this test provided the results of 3.016, 41.227, and 8.092 for CS_{RMS}, $e_{pl,RMS}$ and $e_{pe,RMS}$, respectively.**

***Index Terms*—Full-state Feedback, Furuta Pendulum, Linear Quadratic Regulator, Manual Tuning**

I. Introduction

THE behavior of a dynamic system can be regulated using control system techniques. In this report, a full-state feedback controller will be utilized on a Furuta pendulum system to manipulate its inputs to achieve the desired outcomes.

The Furuta Pendulum

The Furuta pendulum model consists of a freely rotating pendulum that is connected to a motor-drive, rotating platen. Both components of this model will have their own dynamics that will be included in the state-space model when executing an established method of control. The state-space model used, will be comprised of four states, the platen rotation, pendulum rotation, platen angular velocity, and pendulum angular velocity. There will be an assumption that all states are measured, however the angular velocities will just be the numerical derivatives of the rotational positions. Moreover, the model will be characterized with the physical parameters of this pendulum model as ordered similarly to the Cazzolato paper [1]. All the variables will be gathered from direct measurements and in-lab experiments, as shown in (1).

$$\begin{bmatrix}\dot{\theta}_1\\ \dot{\theta}_2\\ \ddot{\theta}_1\\ \ddot{\theta}_2\end{bmatrix} = \begin{bmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ A_{31} & A_{32} & A_{33}-\frac{K_m^2}{R}B_{31} & A_{34}\\ A_{41} & A_{42} & A_{43}-\frac{K_m^2}{R}B_{41} & A_{44}\end{bmatrix}\begin{bmatrix}\theta_1\\ \theta_2\\ \dot{\theta}_1\\ \dot{\theta}_2\end{bmatrix} + \begin{bmatrix}0\\ 0\\ \frac{K_m^2}{R}B_{31}\\ \frac{K_m^2}{R}B_{41}\end{bmatrix}V \tag{1}$$

Manual Tuning

A goal of this lab is to find the optimal performance of the pendulum system through tracking a 0.25Hz square wave that will try to hold a downward position of 0°, with a ±60° tolerance. The first attempt is to use manual tuning to achieve this with an immobilized pendulum and a free-swinging pendulum. This was done with proportional-derivative (PD) control tuning, where K_1 and K_1 act as the proportional control looking to reduce rise time and the steady state error. These values will lead to an increase in percent overshoot, however, with the inclusion of derivative control, K_3 and K_4, the overshoot will be counteracted, as well as the settling time of the system.

Linear Quadratic Regulator

The remaining attempts to determine the optimal performance of the pendulum system will look to be controlled with the Linear Quadratic Regulator (LQR) method. This is an ideal method as it takes the control effort and performance of the system into consideration, where other methods such as the pole-placement only considers the performance. As a result, the tracking and command error should be lower as the approach is more computational.

The quadratic cost function is defined in (2), where the Q portion sets the weight for the errors in the states, the R portion sets the wight on the control where the greater the R the less the system uses the control effort, and lastly, the N portion which will be ignored in this scenario [2].

$$J = x^T(t_1)F(t_1)x(t_1) + \int_{t_0}^{t_1}(x^TQx + u^TRu + 2x^TNu)dt \tag{2}$$

In accordance the feedback control law to minimize the value of the cost (3) is used, where the gain values (K) is given by (4) and the solution of the Riccati differential equation ($P(t)$) is given by (5) [2].

$$u = -Kx \tag{3}$$

$$K = R^{-1}(B^TP(t) + N^T) \tag{4}$$

$$A^T(P(t) + P(t)A - P(t)B + N)R^{-1}(B^TP(t) + N^T) + Q = -\dot{P}(t) \tag{5}$$

II. Methods

A. Hardware Setup

The equipment needed to conduct the lab report and obtain the experimental findings include a Furuta Pendulum assembly with a pendulum encoder, a slip ring, a pendulum driver, and a Pololu DC gearmotor with 30:1 reduction. To record data, a SADI data acquisition device along with a laptop installed with LabVIEW were used to export the data on to a data manipulation software, such as Microsoft Excel.

B. Modeling the Furuta Pendulum

In the previous lab report, the Furuta pendulum system was recreated through a state-space model in MATLAB. This was done by using the Pololu DC gearmotor's specifications which includes the motor resistance (R_m), no-load speed at $V_m = 12\,V$ (ω_m), and motor torque constant (K_m), as well as the mass and inertial properties of the Furuta pendulum.

Then, a velocity step experiment is performed to calculate the DC gain and time constant of the platen base tracking a square wave with a 0.25 Hz frequency. The DC gain (K_{DC}) and time constant (τ) are utilized to calculate the base friction coefficient (b_1) and the system's overall inertia $(\widehat{J_0})$, shown by (6) and (7).

$$b_1 = \frac{K_m}{K_{DC} R_m} - K_m \left(\frac{V_m}{51.45 * \omega_m} \right) \tag{6}$$

$$\widehat{J_0} = \tau \left(\frac{K_m^2 + b_1 R_m}{R_m} \right) \tag{7}$$

Next, a free-swing experiment is performed with the pendulum. With no base movement and using the oscillations of the pendulum between ±15 degrees, the pendulum friction coefficient (b_2) is determined through the envelope function (8) of the pendulum's oscillations and the total rotational inertia about the point where the pendulum rotates about $(\widehat{J_2})$.

$$envelope\ function = A_0 e^{\frac{-b_2}{2\widehat{J_2}}} \tag{8}$$

Finally, with all the required values already obtained, the state-space model is generated in MATLAB along with the impulse response of the platen rotation system and the pendulum rotation system, seen in Fig. 1. The state transition matrix (A), the input matrix (B), the output matrix (C), and the feedforward matrix (D) of the model are shown by (9), (10), (11), and (12), respectively.

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0.4088 & -83.9055 & 0.0698 \\ 0 & -7.3117 & 8.9708 & -1.2483 \end{bmatrix} \tag{9}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ 126.1875 \\ -13.4914 \end{bmatrix} \tag{10}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \tag{11}$$

$$D = [0] \tag{12}$$

C. Manual Tuning

The manually tuning method was initially tested with an immobilized pendulum arm in the downward position, help with adhesive tape. This is to ensure that the platen base could rotate freely while the pendulum remains stationary. The K_1 and K_3 gains are first manually tuned to enable the base to accurately track a square wave with an amplitude of 60 degrees at a frequency of 0.25 Hz. The K_2 and K_4 gains are set to zero during this manual tuning process (Task 1a). After iteratively adjusting the gains to achieve optimal performance, the adhesive tape was removed to allow the pendulum to freely swing. With the same tuned gains, the base was again observed to track the square wave with an amplitude of ±60 degrees at a frequency of 0.25 Hz (Task 1b).

Next, with the pendulum in the downward configuration, gains K_1, K_2, K_3, and K_4 are to be manually tuned to minimize pendulum motion while also allowing the base to effectively track the same square wave stipulated in Task 1. Initially, gains K_1 and K_3 are set to zero to focus on tuning the gains related to the pendulum's rotation, K_2 and K_4. To balance the volatile motion of the pendulum due to the base, negative values are considered in the tuning process of gains K_2 and K_4 (Task 2).

D. LQR Control

Making use of the state-space model of the Furuta pendulum, a full-state feedback controller was designed with a focus on the LQR method. With the goal to fine-tune the performance while also considering control effort, a maximum error was established for the θ_1 and θ_2 states. The maximum error was set to 5 degrees or 0.09 radians. The Q matrix of the controller is determined by thinking of the diagonal values as being the reciprocal of the designated maximum error for each state [2]. For the $\dot{\theta}_1$ and $\dot{\theta}_2$ states, no maximum error is considered into the formulation of the Q matrix. The R matrix of the controller was set to 0.9. Setting the R matrix equal to 0.9 implies that control effort is penalized to some extent but not excessively. It was desired to allow the controller to achieve the desired performance while also minimizing control effort, although allowing slightly high control effort if necessary.

With the chosen Q and R matrix values, the designed gains were found using the lqr() function in MATLAB. All gains are then implemented on the Furuta pendulum system with the base tracking the square wave with an amplitude of 60 degrees at a frequency of 0.25 Hz. The performance of system stabilizing the pendulum oscillations was mainly observed (Task 3).

Adhering to the same process outlined in Task 3, the maximum error for the θ_1 and θ_2 states was halved, making the new maximum error be 2.5 degrees or 0.045 radians. The newly designed gains are then found with MATLAB again and tested on the Furuta Pendulum. The performance of system stabilizing the pendulum oscillations was mainly observed (Task 4a).

As part of the final analysis of the system, the R matrix value was modified, first by a factor of 0.1 and then by a factor of 10. The Q matrix values are kept the same as the ones used in Task 3. Similar to Task 4a, both sets of newly designed gains are determined and tested, first for the R value of 0.09 and then for the R value of 9 (Task 4b).

E. Quantified System Performance

For each task, the Root-Mean-Square (RMS) values are calculated for the command signal (CS), the platen angle error (e_{pl}), and the pendulum angle error (e_{pen}). As shown in (13), the RMS is calculated using the number of measurements (N) and the squared value of each measurement (x_i) in a single full cycle.

$$RMS = \sqrt{\frac{1}{N}\sum_{i}^{N}(x_i)^2} \quad (13)$$

III. Results

A. Impulse Response of Open-Loop Model

In the previous lab report the physical parameters of the rotational pendulum system were characterized and utilized to generate and impulse response time plot of the model in the downward pendulum configuration, as shown in Fig. 1.

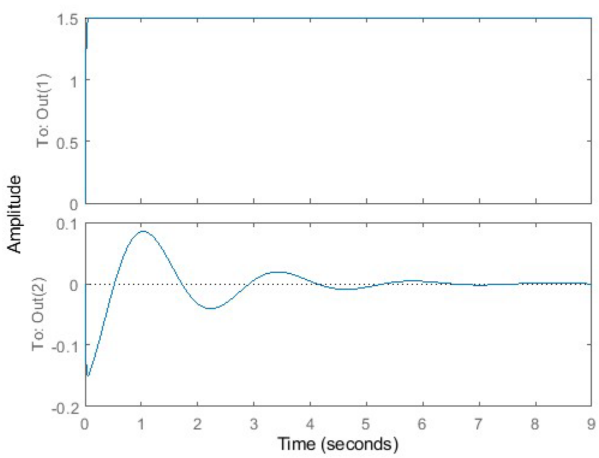

Fig. 1. Impulse response of the platen rotation system on the top and the pendulum rotation of the system [4].

The impulse response of the platen rotation system displays a resulting amplitude of 1.5 indicating rotational displacement on the initial motion was increasing relatively rapid until it was discontinued due to the platen's friction. The impulse response of the pendulum rotation system exhibits a damped oscillatory reaction. The damping occurs due to the pendulum system's frictional forces working in tandem with the added weight at the end of the pendulum.

These responses follow what was taught in the class as the rotational displacement of the pendulum opposes the rotational displacement of the platen in the initial movement of the system in the downward pendulum configuration [2].

B. Feedback Gains Observation

Each task performed as stated in the Methods section made use of different combinations of values for the K_1, K_2, K_3, and K_4 gains. The values for each gain pertaining to certain task are shown in Table I. K_1 and K_2 were the largest (absolute value) for Task 4b where the R matrix value was set to 0.09. K_3 was the largest for both Tasks 1 and 2, where manual tuning was utilized. K_4 was the largest for Task 2.

TABLE I
GAIN VALUES FOR EACH TASK

Task	K_1	K_2	K_3	K_4
1a, 1b	5.5	0.0	0.42	0.0
2	5.5	-0.2	0.42	-8.0
3	3.513	-0.144	0.051	0.097
4a	4.969	-0.171	0.074	0.159
4b (0.1R)	15.713	-0.246	0.233	0.615
4b (10R)	1.571	-0.074	0.021	0.023

C. RMS tracking errors and RMS command effort

RMS tracking errors and command effort reveal the system's performance. In Tasks 1a and 1b, although the RMS tracking base error is relatively moderate at around 12.7, the RMS tracking pendulum error is notably high at 83.12 for Task 1b. Conversely, Task 1b demonstrates a lower command effort of 2.89 than Task 1a. Task 2 strikes a balance with a moderate tracking pendulum error of 8.09 and a low command effort of 3.01563, although the tracking base error is relatively high. Task 3 makes a balance between command effort and tracking base error, although the tracking pendulum error of 41.22 is the second worst. Task 4a, similar to Task 2, exhibits a high tracking base error but maintains low command effort of 4.25. Furthermore, Tasks 4b and 4b highlight the trade-off between tracking accuracy and control effort, with varying penalties on control effort leading to distinct changes in tracking performance and control effort requirements.

TABLE II
RMS CALCULATIONS FOR COMMAND SIGNAL AND ERROR

Task	CS_{RMS}	$e_{pl,RMS}$	$e_{pen,RMS}$
1a (w/ tape)	4.23	12.77	0.75
1b (w/o tape)	2.89	12.60	83.12
2	3.01	41.22	8.09
3	4.14	40.10	69.21
4a	4.25	38.16	8.74
4b (0.1R)	10.59	25.98	26.64
4b (10R)	1.59	44.02	31.61

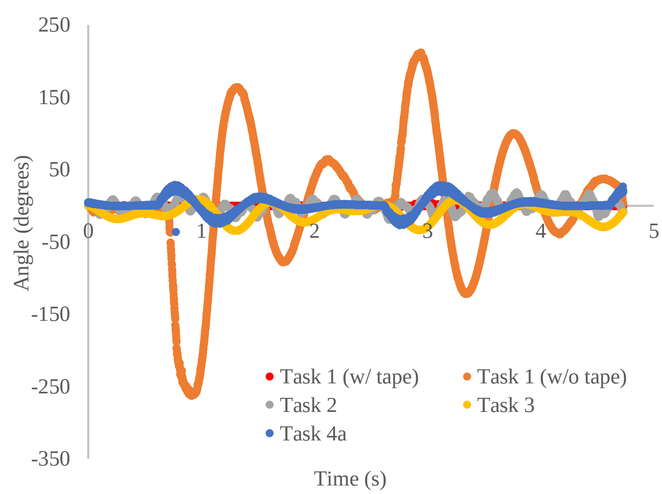

Fig. 2. Pendulum Rotation graphed against time for Tasks 1, 2, 3, 4a.

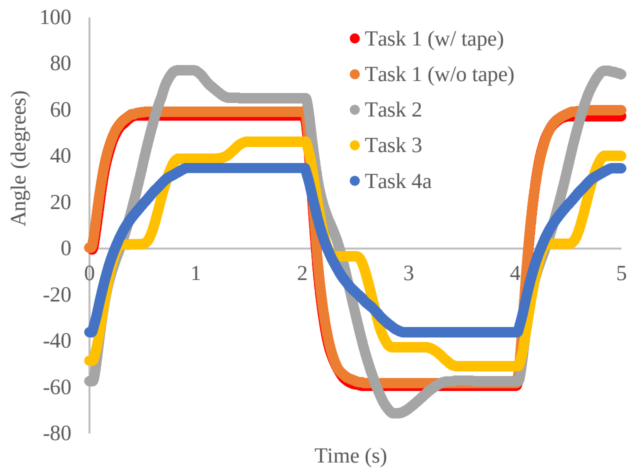

Fig. 2. Platen Rotation graphed against time for Tasks 1, 2, 3, 4a.

IV. Discussion

The purpose of this lab report was to design a full-state feedback controller, to then investigate and evaluate the performance of manually tuned system and a model-based computed system in order to determine the optimal gains for a rotating Furuta pendulum system. A standard applied to all tests and tasks involved trying to keep the pendulum in a downward position with the rotating platen, with a tolerance of ±60° receiving square waves with a frequency of 0.25Hz. The gains were obtained through the manual tuning and LQR methods, where the error of the platen angle and pendulum angle, and the command signal were calculated through RMS and compared.

The optimal method is selected through the error of the pendulum rotation, $e_{pen,RMS}$, as a downward position is desired form the system. It is important to find $e_{pl,RMS}$, or the error of the platen rotation as the lower the magnitude of error, the more stability the system has. These two calculations of error support the command signal of the system because it numerically displays how much effort the control system requires to achieve the desired result. Taking all these into consideration, it can be concluded that Task 2 provides the best setup of a full-state feedback controller regarding the goal of maintaining a free-swinging pendulum with a command signal of 3.01, a platen rotation RMS error of 41.22 and a pendulum rotation RMS error of 8.09.

A. Manual Tuning vs LQR

The overall resulting performance for both the manually tuned system and the model-based computed system displayed positive results. They were able to achieve the task of maintaining the pendulum in a downward position with a tolerance of ±60°. Moreover, through visual inspection of Fig. 2, it can be seen the oscillations for both methods failed to show any trend indicating that there was a superior method, therefore the calculated RMS of the pendulum error would provide the strongest evidence of the optimal method for the pendulum system.

The strongest case for manual tuning came from the Task 2 which involved the tuning of the gains for all the states. The resulting values showed to have a command signal of 3.01, a platen rotation error of 41.22, and a pendulum rotation error of 8.09. For LQR, the best case came from Task 4a which focused on changing the desired poles to achieve smaller pendulum deviations. The calculated results provided a command signal of 4.25, a platen rotation error of 38.16, and a pendulum error 8.74.

The pendulum rotation error itself proves manual tuning is better for achieving the desired results, however this came with a negative effect on the platen rotation error where the LQR method completes the opposite. This becomes irrelevant as the command signal that was determined showed to be relatively much smaller indicating that the overall system had to use less effort to achieve basically the same result. Furthermore, it is significant to note that this manually tuned task could have been further improved if the tuning was not stopped as soon as the results ended up in the desired tolerance of ±60°.

B. Alterations to the LQR Method

Multiple iterations of the LQR method were applied within this report to find the best full-state feedback controller. Task 3 displayed the original LQR method to determine the optimal gains for keeping the pendulum in the downward position. These gains were 3.513, -0.144, 0.051, and 0.097 for K_1, K_2, K_3, and K_4 respectively. This provided relatively poor results, however when the Q values were changed to achieve a smaller pendulum angle deviation the error in the pendulum significantly decreased. These changes only increased the gains of the system slightly giving the values of 4.969, -0.171, 0.074, and 0.159 for K_1, K_2, K_3, and K_4 respectively. This translated to a slightly higher command signal and an overall increase in stability, indicating it to be the optimal choice regarding LQR. The changing of R showed to either greatly increase or decrease the gains and its subsequent errors indicating that changing the weight of the control negatively affects the system and its goals.

C. Recommendations

As with any experimental analysis of state-space models, improvements can be made on the hardware and the methods of analysis to improve the closed-loop performance of the pendulum. To decrease the hardware error, a more stable and secure setup would be desired. This would include a stronger motor, a quicker and more accurate sensor, and a system that accepts more power to allow the rise of the command effort.

Regarding the methodology, an observer with a Kalman filter can be included to allow for the estimation of the system's unmeasured states which will then be compared with measured results to minimize the noise of the system. As a result, more precise results would be available allowing for more accurate actions to come from the control system [3].

References

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